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Turbulent Rayleigh shear flow

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Townsend has derived a relation between mean vorticity and Reynolds stress valid in the wall layer of a turbulent flow. The vorticity Ω appears as a function of the local stress τ and its gradient. Such a relation is better suited for use in the vorticity equation than in the momentum equation. The Rayleigh problem, whose vorticity equation is simply $\partial\Omega/\partial t = \partial^2\tau/\partial y^2$, is introduced as a setting for Townsend's theory. Certain wall-speed programmes are shown to generate Rayleigh layers that are exactly self-similar in the fully turbulent part of the flow. Those layers correspond to Clauser's equilibrium boundary layers. A formal analogy between the two families is found; the analogy becomes quantitatively exact in the limit of infinite Reynolds number. The Rayleigh problem is posed in similarity form. A composite non-linear, ordinary differential equation for the stress profile is deduced from a two-layer model incorporating Townsend's relation for the wall layer and Clauser's constant eddy-viscosity assumption for the outer layer. The profile depends on the wall-speed programme selected and on two empirical constants: the combination $\lambda = \surd(k)/\kappa$ of Clauser's k and Kármán's κ, and Townsend's constant B. Closed-form solutions for arbitrary λ and B are obtained in two important cases: constant wall stress, analogous to constant pressure above a boundary layer, and zero wall stress, corresponding to continuous separation. The velocity profile in the wall region of a continuously separating Rayleigh layer is found to depend sensitively on B.

1. Introduction

Stress and velocity are connected by the relation $\tau = \nu\,\partial u/\partial y$ in laminar boundary layers, and there is no reason to consider one more fundamental than the other. Reynolds stress and velocity distributions have been sharply distinguished in turbulent boundary-layer theory, however, not so much because one of them is more deeply connected with the underlying structure of the turbulence, but because the stress varies slowly and simply near a rigid wall, whereas the velocity profile exhibits a striking logarithmic behaviour that demands explanation. The Prandtl mixing-length theory provides an explanation in the form

$$\Omega = \tau^{\frac{1}{2}}/\kappa y, \tag{1}$$

where $\kappa \approx 0{\cdot}41$ is Kármán's constant. Co-ordinates x, y, z are chosen so that x specifies distance downstream and y specifies distance perpendicular to the wall.

Also

$U, V, 0$ are the corresponding mean velocity components,
u, v, w are the turbulent fluctuations,
P is the mean kinematic pressure,
p is the fluctuation pressure,
$\Omega = \partial U/\partial y$ is the mean vorticity, to the boundary layer approximation, and
$\tau = -\overline{uv}$ is the Reynolds stress.

As long as τ approaches some constant near the wall, the logarithmic behaviour of U is enforced by (1) regardness of any constraint imposed by the mean momentum and continuity equations:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{dP}{dx} = \frac{\partial \tau}{\partial y}, \tag{2}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{3}$$

A complicated balance between non-linear advection and viscous diffusion determines the shape of a laminar velocity profile. In the turbulent case, the velocity near the wall is governed primarily by a balance of random processes represented in (1).

Provided τ varies slowly near the wall, the $1/y$ factor in (1) determines the behaviour of Ω. It is not obvious, however, that equations (1) and (2) are consistent with a non-singular stress profile. Until recently, the smoothness of the stress distribution had to be taken as an experimental fact rather than an analytical consequence of the mixing-length theory and the momentum equation.

Clauser (1956) drew attention to the problem of mean dynamics. He distinguished a family of boundary layers having defect profiles $(U_\infty - U)/u_\tau$ nearly similar at successive stations downstream; U_∞ is the free-stream velocity and u_τ is the square root of the local wall stress. These layers are characterized by a constant ratio between the wall stress and the pressure force across the displacement thickness of the layer. Clauser solved the similarity form of equations (2), (3) in an outer layer only, assuming that beyond some distance from the wall τ is related to Ω by an eddy viscosity $\nu_e \equiv \tau/\Omega$ independent of y. He did not extend the momentum balance into the wall layer. Mellor & Gibson (1966) completed the programme by presenting numerical solutions for the whole family of 'equilibrium' boundary layers. They assume that τ and Ω are related by (1) in the wall layer and by Clauser's constant eddy viscosity in the outer layer. They fix the junction between the layers at the point where the two relations are equivalent. They express the momentum equation in terms of the velocity profile and assign no primacy to stress. No difficulty arises, because (1) can be written in the form $\tau = \tau(\Omega)$.

Townsend (1961) derived the relation

$$\Omega = \frac{\tau^{\frac{1}{2}}}{\kappa y}\left(1 - B\frac{y}{\tau}\left|\frac{\partial \tau}{\partial y}\right|\right) \tag{4}$$

as an improvement on (1). Equation (4) describes the mean flow in the wall regions of both the constant pressure boundary layer and the continuously separating layer of Stratford (1959) if $B \approx 0{\cdot}2$. Townsend's relation is ill suited to the approach of Mellor & Gibson, because it cannot be inverted into a local relation $\tau = \tau(\Omega)$. $\Omega\{\tau\}$ appears instead as a local function of τ and its first derivative. The relation suggests that Reynolds stress should be taken as the primary dynamic quantity in accord with the older approaches. The boundary-layer *vorticity* equation,

$$U\frac{\partial \Omega}{\partial x} + V\frac{\partial \Omega}{\partial y} = \frac{\partial^2 \tau}{\partial y^2},$$

obtained by differentiating (2) with respect to y and using (3), establishes the dynamic connexion between Ω and τ. A comprehensive $\Omega\{\tau\}$ relation (e.g. Townsend's equation (4) in the wall layer and Clauser's constant eddy-viscosity

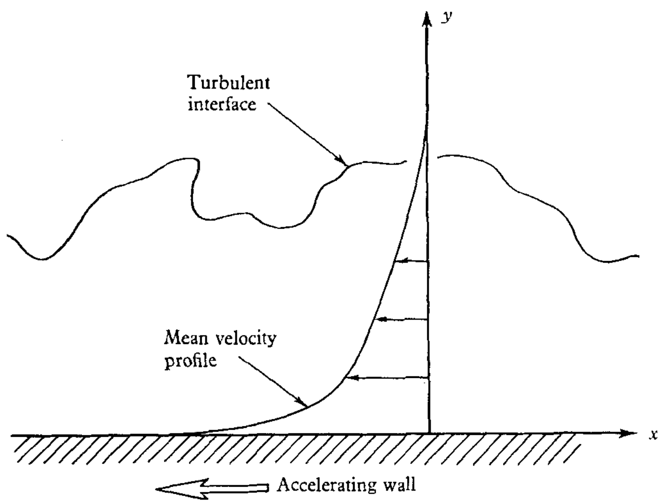

FIGURE 1. Turbulent Rayleigh flow.

assumption in the outer layer) can be substituted into the mean vorticity equation to produce an equation for τ. The equation is complicated, since U and V have to be expressed as integrals of Ω.

The present paper is a study of a simpler problem that contains all the physics of turbulence bound in Townsend's relation (4)—Rayleigh shear flow. Rayleigh flow is sketched in figure 1. An infinite rigid plate covering the (x, z)-plane accelerates in the negative x-direction (a negative wall speed is chosen so that stress comes out positive). The incompressible turbulent flow above the plate is statistically homogeneous in x and z. Averages can be taken over planes parallel to the (x, z)-plane. Mean velocity and Reynolds stress are nonsteady, depending on y and t. The continuity equation is identically satisfied with $V = 0$, and the mean momentum equation is

$$\frac{\partial U}{\partial t} = \frac{\partial \tau}{\partial y}. \tag{5}$$

Thus

$$\frac{\partial \Omega}{\partial t} = \frac{\partial^2 \tau}{\partial y^2}. \tag{6}$$

Once an $\Omega\{\tau\}$ relation is specified, the vorticity equation (6) can be written as a self-contained differential equation for the Reynolds stress. The natural boundary condition is an imposed *stress* history rather than a velocity history. Wall-stress histories will be found that admit scale similarity between the stress-bearing turbulence and the mean vorticity in the fully turbulent part of the flow. The Rayleigh layers generated by such histories correspond to Clauser's equilibrium boundary layers, and the correspondence is surprisingly close.

2. Analogy between equilibrium Rayleigh and boundary layers

Laminar Rayleigh flow is often used as an analogy to clarify the way vorticity spreads in a boundary layer. The analogy for equilibrium turbulent flows runs much deeper. The Rayleigh momentum equation is linear in the velocity profile. The turbulent boundary-layer equation behaves almost as if it too were linear, because the non-linear advection terms are significant only near the wall where they are swamped anyway by the rapid drop in eddy resistance. Each similarity solution of the Rayleigh problem is, as we shall see, identical to an equilibrium boundary-layer solution at *infinite Reynolds number*.

Consider, for example, the constant pressure boundary layer. The notation and technique of Mellor & Gibson (1966) will be followed except that tildes will be used over dimensionless variables to distinguish them from variables scaled in a slightly different way in subsequent sections. If the mean velocity defect and the eddies adjust to the same local velocity scale $u_\tau(x)$ and length $\Delta(x)$, then the defect and stress profiles have the forms

$$U_\infty - U = u_\tau \tilde{f}'(\tilde{\eta}),$$

$$\tau = u_\tau^2 \tilde{s}(\tilde{\eta}),$$

where

$$\tilde{\eta} = \frac{y}{\Delta}.$$

Mellor & Gibson choose u_τ^2 to be the wall stress and set

$$\Delta = \int_0^\infty \frac{U_\infty - U}{u_\tau}\, dy$$

so that $(u_\tau/U_\infty)\,\Delta = \delta^*$, the displacement thickness of the boundary layer. If the stream function $\tilde{f} = 0$ at $\tilde{\eta} = 0$, then those choices establish the boundary conditions

$$\left.\begin{aligned} \tilde{f}(0) = 0, \quad \tilde{f}(\infty) = 1, \\ \tilde{s}(0) = 1, \quad \tilde{s}(\infty) = 0. \end{aligned}\right\} \tag{7}$$

The similarity forms can be substituted into equations (2) and (3) to find the relation between $\tilde{s}$ and $\tilde{f}$ based on the mean dynamics. The result involves x-derivatives of u_τ, but they can be eliminated by the law of the wall:

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\frac{u_\tau y}{\nu} + C_1,$$

in the region $\delta^* \gg y \gg \nu/u_\tau$; ν is the molecular viscosity and $C_1 \approx 4{\cdot}9$ is a universal constant. If the $\Omega\{\tau\}$ relation assumed is reasonable (e.g. equation (1) or (4)), the solution to the dynamic problem approaches

$$\frac{U_\infty - U}{u_\tau} \to \tilde{C}_2 - \frac{1}{\kappa} \ln \frac{y}{\Delta}$$

as $\tilde{\eta} \to 0$; Mellor & Gibson find $\tilde{C}_2 \approx -0{\cdot}6$ for the constant pressure layer. Thus u_τ and Δ are related by

$$\frac{1}{\gamma} = \frac{1}{\kappa} \ln \frac{\Delta U_\infty}{\nu} \gamma + C_1 + \tilde{C}_2, \tag{8}$$

where $\gamma \equiv u_\tau / U_\infty$. The argument of the logarithm can be written $U_\infty \delta^* / \nu$, the Reynolds number of the boundary layer based on displacement thickness. γ is small at large Reynolds numbers, typical experimental values being 0·02–0·05. The equation governing the mean dynamics for $dP/dx = 0$ is

$$\tilde{s}' - \frac{1}{\gamma} \frac{d\Delta}{dx} \tilde{\eta} \tilde{f}'' = \frac{d\Delta}{dx} \left\{ \frac{1}{\kappa + \gamma} \tilde{f}' + \frac{\gamma}{\kappa + \gamma} (\tilde{f}\tilde{f}'' - \tilde{f}'^2) - \tilde{f}\tilde{f}' \right\}. \tag{9}$$

A differential equation for $\Delta(x)$ is found by integrating (9) over all $\tilde{\eta}$ and using the boundary conditions (7).

If the right-hand side of (9) is negligible, the integration gives

$$d\Delta/dx \approx \gamma \tag{10}$$

so that

$$\tilde{s}' - \tilde{\eta} \tilde{f}'' \approx 0. \tag{11}$$

Equation (11), involving no functions of x, is the kind of similarity relation one would wish for. Equation (8) can be written in the form $\Delta = \Delta(\gamma)$ and substituted into (10). The resulting differential equation for γ can be integrated to find the wall stress for all x.

Let us leave open for a moment the question whether the right-hand side of (9) really is negligible and consider the analogous Rayleigh problem. The velocity defect is replaced with

$$-U/u_\tau = \tilde{f}'(\tilde{\eta}),$$

the scale

$$\Delta = -\int_0^\infty \frac{U}{u_\tau} dy$$

is a function of time, but the meaning of $\tilde{s}$ and the boundary conditions (7) are unchanged. Suppose the velocity of the moving plate is adjusted so that *the wall stress remains constant in time*. Then u_τ is constant, and the momentum equation (5) becomes

$$\tilde{s}' - \frac{1}{u_\tau} \frac{d\Delta}{dt} \tilde{\eta} \tilde{f}'' = 0.$$

An integration from $\tilde{\eta} = 0$ to ∞ yields $d\Delta/dt = u_\tau$, so

$$\Delta = u_\tau (t - t_0), \tag{12}$$

where t_0 is the virtual origin in time when the Rayleigh layer would have had zero thickness, had it followed a similar development throughout its history. Thus

$$\tilde{s}' - \tilde{\eta} \tilde{f}'' = 0, \tag{13}$$

exactly what the boundary-layer equation would be if the non-linear (and non-similar) right-hand side could be dropped. The Rayleigh velocity profile for constant wall stress is identical to the universal part of the constant pressure boundary-layer profile. In particular,

$$-\frac{U}{u_\tau} \to \tilde{C}_2 - \frac{1}{\kappa}\ln\frac{y}{\Delta}$$

as $\tilde{\eta} \to 0$, where $\tilde{C}_2$ can be taken from a boundary-layer solution. The law of the wall, in the form

$$\frac{U-U_w}{u_\tau} = \frac{1}{\kappa}\ln\frac{u_\tau y}{\nu} + C_1, \tag{14}$$

should hold for a wall moving at a speed U_w. The wall-speed programme

$$U_w = u_\tau\left\{\frac{1}{\kappa}\ln\frac{u_\tau^2(t-t_0)}{\nu} + C_1 + \tilde{C}_2\right\} \tag{15}$$

therefore generates a constant stress Rayleigh layer whose depth grows linearly with time. The mean-field and eddy scales are exactly similar in the fully turbulent part of the flow, and the stress and velocity profiles are related therein by a similarity equation (13) very much like the equation for the constant pressure boundary layer. The differences are that the complete boundary-layer equation (9) contains irremovable functions of x, that boundary-layer profiles can therefore not be exactly similar, and that the relation of u_τ to x in the boundary-layer problem is much more involved than the analogous relation of U_w to t in the Rayleigh problem.

Equation (11) is a dimensionless version of the linearized momentum equation

$$U_\infty\frac{\partial \mathscr{U}}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \tag{16}$$

for the velocity defect $\mathscr{U} = U_\infty - U$. Equation (11) follows directly from (16) if u_τ is assumed to vary slowly. Equation (16) is valid in the outer part of the layer where $\mathscr{U} \ll U_\infty$. Close to the wall, where $\mathscr{U} \sim U_\infty$ and equations (16) and (11) are inaccurate, the flow is dominated by a rapid fall in eddy resistance. As the Reynolds number $U_\infty \delta^*/\nu$ becomes large, γ becomes small, and the fractional velocity defect $\mathscr{U}/U_\infty = \gamma \tilde{f}'(\tilde{\eta})$ becomes small at any fixed $\tilde{\eta}$. Equation (16) becomes valid everywhere in the layer except in a region of decreasing thickness in $\tilde{\eta}$, just above the laminar sublayer, where the flow is insensitive to non-linear advection anyway. It is not surprising that the constant pressure boundary-layer profile tends to the shape of a Rayleigh flow at large Reynolds numbers: the quantity x/U_∞ in (16) plays the part that t plays in (5).

The existence of functions of x on the right-hand side of (9) means that the boundary layer cannot achieve full similarity at finite Reynolds numbers, even in the fully turbulent part of the flow. The boundary layer responds to both U_∞ and u_τ, and the ratio of the two is not fixed. U_∞ is defined with respect to the wall and comprises the velocity slip that occurs in the laminar sublayer adjacent to the wall. That slip depends on the molecular viscosity ν and does not scale on the velocity u_τ characterizing the turbulent eddies. The fully turbulent part of a

Rayleigh layer, on the other hand, never sees the wall speed U_w. Even though U_w depends on ν and is non-similar, the fully turbulent part of a Rayleigh layer attains exact similarity if *the right amount of stress is transmitted through the laminar sublayer.*

One such stress programme is constant wall stress, as we have seen, and the resulting Rayleigh flow has the same dynamics as a constant pressure boundary layer in the limit of infinite Reynolds number. A family of such programmes is studied in the next few sections. Each member of the family corresponds to one of Clauser's equilibrium boundary layers. The similarity equation for the boundary layers (Mellor & Gibson's equation (43 *a*)) can be obtained from the general linearized momentum equation

$$U_\infty \frac{\partial \mathscr{U}}{\partial x} + \mathscr{U} \frac{dU_\infty}{dx} - y \frac{dU_\infty}{dx} \frac{\partial \mathscr{U}}{\partial y} + \frac{\partial \tau}{\partial y} = 0,$$

which comes from (2), (3), and the free-stream condition $dP/dx = -U_\infty dU_\infty/dx$. Equilibrium layers have constant values of the pressure gradient parameter

$$\beta = \frac{\delta^*}{u_\tau^2} \frac{dP}{dx} = -\frac{\Delta}{u_\tau} \frac{dU_\infty}{dx}$$

and satisfy the similarity equation

$$\tilde{s}' - (1+2\beta)\tilde{\eta}\tilde{f}'' - 2\beta\tilde{f}' = 0 \qquad (17)$$

at infinite Reynolds number.

3. The family of self-similar Rayleigh flows

The vorticity equation (6) admits solutions of the form

$$\tau = \sigma(t)\, s(\eta),$$

$$\Omega = \frac{[\sigma(t)]^{\frac{1}{2}}}{l(t)} g(\eta),$$

$$\eta = \frac{y}{l(t)},$$

provided either $\sigma \sim e^{\alpha(t-t_0)}, \quad l \sim e^{\alpha(t-t_0)/2},$

or $\sigma \sim (t-t_0)^c, \quad l \sim (t-t_0)^{1+c/2}.$

Either combination yields an equation free of explicit dependence on time; as far as the mean dynamics is concerned, the stress-bearing eddies and the mean flow can adjust to the same velocity and length scales. Therefore they do so, according to the arguments of Townsend (1956).

An algebraic combination with $c > -2$ represents a spreading Rayleigh layer with zero thickness at a finite virtual origin in time. The other algebraic combinations of σ and l represent contracting layers. Boundary layers actually can contract under highly favourable pressure gradients: V is then strongly negative above the wall, and inward convection by the mean field overcomes the tendency of turbulence to spread. No such convection mechanism is available in a Rayleigh flow, so the cases $c < -2$ must be ruled out. The case $c = -1$ corresponds to continuously separating flow with zero wall stress. The cases $-2 < c < -1$, involving negative wall stress and various degrees of reversed flow, are excluded from the discussion of § 4 *et seq.* to avoid cumbersome absolute value signs.

It is convenient to write

$$\left.\begin{aligned} \sigma &= Q(t-t_0)^c, \\ l &= (kQ)^{\frac{1}{2}}(t-t_0)^{1+c/2}, \end{aligned}\right\} \tag{18}$$

and to set the wall stress $$u_\tau^2 = (1+c)\,\sigma. \tag{19}$$

Equation (6) becomes $$s'' + k^{\frac{1}{2}}(1+\tfrac{1}{2}c)\,\eta g' + k^{\frac{1}{2}} g = 0, \tag{20}$$

and the boundary conditions on stress are

$$s(0) = 1+c, \quad s(\infty) = 0. \tag{21}$$

Q is a measure of the intensity of the stress imposed at the plate, and k is a dimensionless constant whose value is specified in § 4. An alternative procedure is to follow Mellor & Gibson and set $\sigma = u_\tau^2$ and $l = \Delta$ as was done in the previous section. The advantage of the present choice for σ is that a transformation for highly retarded flows, which Mellor & Gibson have to carry out explicitly, is made unnecessary by the boundary condition $s(0) = 1+c$. The advantage of introducing k at this stage is explained in § 4.

Equations (20) and (21), together with a relation $\Omega\{\tau\}$ written in similarity form $g\{s\}$, determine the stress distribution s. The velocity profile can be found directly from s by integrating the momentum equation (5) from t_0 to t. The result in similarity form is

$$-f'(\eta) = \frac{U}{\sigma^{\frac{1}{2}}} = \frac{1}{k^{\frac{1}{2}}}\frac{\eta^{d-1}}{1+c/2}\int_\eta^\infty \frac{s'(\zeta)}{\zeta^d}\,d\zeta, \tag{22}$$

where $$d = \frac{1+c}{1+c/2}$$

and $-f'' = g$. f' can also be expressed as an integral of $g\{s\}$, but the two modes of expression must be the same numerically as long as the dynamic relation (20) between s and g is satisfied everywhere. For a reasonable assumption about $\Omega\{\tau\}$,

$$f' \to C_2(c) - \frac{(1+c)^{\frac{1}{2}}}{\kappa}\ln\frac{y}{l}$$

as $\eta = y/l \to 0$. A comparison with the law of the wall (14) shows that the wall-speed programme

$$U_w = -\sigma^{\frac{1}{2}}\left\{\frac{(1+c)^{\frac{1}{2}}}{\kappa}\ln\left((1+c)^{\frac{1}{2}}\frac{\sigma^{\frac{1}{2}}l}{\nu}\right) + (1+c)^{\frac{1}{2}}C_1 + C_2(c)\right\} \tag{23}$$

is required to generate the self-similar Rayleigh layer whose stress and length scales satisfy equations (18).

The derivative of (17) is structurally the same as equation (20) if

$$\beta(c) = -\frac{\frac{1}{4}c}{1+c},$$

the remaining differences being removable by rescaling s, g and η to conform with the scaling adopted in §2. A solution of (20) can be rescaled as the stress profile of the equilibrium boundary layer characterized by $\beta(c)$. The boundary-layer and Rayleigh solutions, of course, share the same assumption about $\Omega\{\tau\}$. Details of the connexion between equilibrium boundary and Rayleigh layers are treated elsewhere (Crow 1966).

4. $\Omega\{\tau\}$ based on the two-layer model

The object now is to find a relation of the form $g\{s\}$ reasonably well founded on physical arguments about the stress-bearing eddies. Arguments originally advanced in boundary-layer theory will be adapted to the Rayleigh problem. The two-layer model of Clauser and Townsend will be used, with a wall layer in energy equilibrium and an outer layer having an eddy viscosity independent of y.

Townsend's (1961) energy balance argument can be taken over with little alteration to establish (4) near the wall of a Rayleigh layer. The turbulent energy equation for the Rayleigh problem is

$$\frac{\partial E}{\partial t} - \tau\Omega + \frac{\partial F}{\partial y} + \epsilon = 0,$$

where

$E = \frac{1}{2}\overline{(u^2+v^2+w^2)}$ is the turbulent energy density,

$F = \overline{v\{p+\frac{1}{2}(u^2+v^2+w^2)\}}$ is the lateral flux of turbulent energy, and

$\epsilon \equiv E^{\frac{3}{2}}/L_\epsilon$ is the rate of energy dissipation.

L_ϵ is a dissipation length defined in terms of ϵ as shown. Energy equilibrium in a Rayleigh flow means that

$$|\partial E/\partial t| \ll \tau\Omega, \tag{24}$$

where $\tau\Omega$ is the rate of generation of turbulent energy by interaction with the mean field. Townsend's relation (4) follows provided that E, F and τ scale on the same velocity, so

$$\tau = 2a_1 E,$$

for example, and that $L_\epsilon \sim y$ near the wall. The similarity form of (4) is

$$g = (s^{\frac{1}{2}}/\kappa\eta)\, b, \tag{25}$$

where

$$b \equiv 1 - B(\eta/s)|s'|.$$

$b = 1$ gives the Prandtl result (1); the additional term in b represents the effect of lateral transport of turbulent energy. There is no reason to doubt that the values $\kappa \approx 0{\cdot}41$ and $B \approx 0{\cdot}2$ obtained from experiments on boundary layers hold for Rayleigh flows as well.

Inequality (24) can be written

$$\frac{1}{2a_1}\left|\frac{\partial \tau}{\partial t}\right| \ll \tau\Omega.$$

The vorticity equation (6) implies that

$$\frac{\partial \tau}{\partial t} = 2\kappa y \tau^{\frac{1}{2}} \frac{\partial^2 \tau}{\partial y^2}$$

wherever the Prandtl relation (1) is an adequate approximation. The condition for energy equilibrium is therefore

$$\frac{\kappa^2}{a_1} y^2 \left|\frac{\partial^2 \tau}{\partial y^2}\right| \ll \tau,$$

or, in similarity form,

$$\frac{\kappa^2}{a_1} \eta^2 |s''| \ll s. \tag{26}$$

a_1 is known to be about 0·15 in boundary layers (Bradshaw, Ferriss & Atwell 1967), so $\kappa^2/a_1 \approx 1$. Inequality (26) is strongly satisfied in the wall regions of calculated stress profiles. The requirement $L_\epsilon \sim y$ fails well within the energy-equilibrium region and constitutes the most important limitation on (4).

The $\Omega\{\tau\}$ relation in the outer layer is based on an assumption about the eddy viscosity $\nu_e \equiv \tau/\Omega$. Clauser (1956) found that the turbulent Reynolds number

$$k^{-1} = U_\infty \delta^* / \nu_e$$

is very nearly a universal constant in the outer parts of equilibrium boundary layers. k is about 0·015. The eddy viscosity is thus

$$\nu_e = k \int_0^\infty (U_\infty - U)\, dy.$$

The analogous assumption for the outer part of a self-similar Rayleigh layer is

$$\nu_e = -k \int_0^\infty U\, dy.$$

Therefore

$$\Omega = \tau / kM,$$

where $M = -\int_0^\infty U\, dy$ is the negative of the total mean momentum in the flow. According to the momentum equation (5) and the relations (18) and (19),

$$M = \int_{t_0}^{t} u_\tau^2(T)\, dT = Q(t - t_0)^{1+c}, \tag{27}$$

so the similarity form of the constant-k assumption is

$$g = s / k^{\frac{1}{2}}. \tag{28}$$

The comprehensive $g\{s\}$ relation consists of equations (25) and (28) joined at their point of equality η_e. It is convenient to define the dimensionless constant

$$\lambda \equiv k^{\frac{1}{2}} / \kappa \approx 0{\cdot}30$$

and to write

$$r \equiv s^{\frac{1}{2}}.$$

Expressions (25) and (28) give equal values of g where

$$\eta_e r_e = \lambda b_e. \tag{29}$$

That defines the junction between wall and outer layers. In the wall layer $0 \leqslant \eta \leqslant \eta_e$, equation (20) takes the form

$$s'' = \tfrac{1}{2}\lambda \left[c\frac{(br)}{\eta} - (2+c)\,(br)' \right], \tag{30}$$

a non-linear, ordinary differential equation for the stress. In the outer layer $\eta > \eta_e$,

$$s'' + (1+\tfrac{1}{2}c)\,\eta s' + s = 0. \tag{31}$$

Equations (30) and (31) must be solved separately and the solutions matched at the point η_e determined by (29). The single boundary condition $s(0) = 1+c$ is applied to the wall solution, and $s(\infty) = 0$ is required of the outer solution. Since both equations (30) and (31) are of second order, two matching conditions are required. One of them is continuity of s; it is easy to show the other is then continuity of slope s'. Equations (30) and (31), taken together, have the form

$$s'' = H(\eta, s, s'; \lambda),$$

where H changes its functional form at η_e. Since s is continuous, the most s' can do is jump. Consequently H and hence s'' can have at most a jump discontinuity at η_e. Hence s' is continuous.

None of the equations (29)–(31) involves κ or k separately, but only combined into λ. Under the present assumptions about $\Omega\{\tau\}$, the stress distribution for any c depends on two empirical constants only: λ and B. The constant k was introduced in equations (18) to bring out that fact. k fixes the relationship between the depth and the momentum content of a Rayleigh layer, but it does not separately affect the shape of the stress profile.

5. General consequences of the two-layer model

The outer equation (31) is well known (Townsend 1956, p. 270; Jeffreys & Jeffreys 1956, p. 622). It can be written in standard form

$$s_{\zeta\zeta} + \zeta s_{\zeta} - ns = 0,$$

with $$\zeta = (1+\tfrac{1}{2}c)\,\eta$$

and $$n = -(1+\tfrac{1}{2}c)^{-1}.$$

The solution vanishing as $\zeta \to \infty$ is written

$$s = A\,Hh_n(\zeta).$$

Some special Hh_n functions are

$$\left.\begin{aligned} Hh_{-1}(\zeta) &= e^{-\frac{1}{2}\zeta^2}, \\ Hh_{-2}(\zeta) &= \zeta e^{-\frac{1}{2}\zeta^2}. \end{aligned}\right\} \tag{32}$$

For η very small, the wall-layer equation (30) can be treated generally by means of the transformation $\xi = 1/\eta$. The resulting equation for $s(\xi)$ is

$$s_{\xi\xi} + \frac{2}{\xi}s_{\xi} = \tfrac{1}{2}\lambda \left[\frac{c}{\xi^3}\,(br) + \frac{(2+c)}{\xi^2}\,(br)_{\xi} \right].$$

As $\xi \to \infty$, $s(\xi)$ smoothly approaches its asymptotic value $(1+c)$. It will be found that $b \to 1$ except in the case $c = -1$. Thus

$$s_{\xi\xi} + \frac{2}{\xi} s_\xi \to \frac{\lambda c(1+c)^{\frac{1}{2}}}{2\xi^3} + O\left(\frac{\lambda s_\xi}{\xi^2(1+c)^{\frac{1}{2}}}\right)$$

as $\xi \to \infty$, and the second term on the right becomes negligible compared with the second term on the left. The solution valid for large ξ is

$$s(\xi) \to (1+c) + \frac{a}{\xi} - \frac{\lambda c\,(1+c)^{\frac{1}{2}}}{2} \frac{\ln \xi}{\xi},$$

so
$$s(\eta) \to (1+c) + a\eta + \frac{\lambda c(1+c)^{\frac{1}{2}}}{2} \eta \ln \eta \tag{33}$$

as $\eta \to 0$. The constant a can be determined only by solving the complete problem. s' has a logarithmic singularity at $\eta = 0$ unless $c = 0$ or -1, but it is easy to see from (25) that $b \to 1$ at the origin anyway. The argument leading to (33) breaks down when $c = -1$, but the result is, in fact, valid in that case too.

The non-analytic behaviour of s near the origin is a surprising consequence of the mean flow dynamics. Mellor (1966) found the same kind of behaviour in the analogous boundary-layer stress profiles. The singularity is not strong enough to alter the logarithmic part of the velocity profile, and it is absent if the wall stress is constant, $c = 0$, or is zero, $c = -1$. Exact solutions of (30) have been found in those two cases for arbitrary λ and B. The solutions make possible a complete analytic description of the $c = 0$ and $c = -1$ Rayleigh layers. The absence of the $\eta \ln \eta$ term near the wall suggests that the solutions for $c = 0$ and -1 may be especially simple. Equation (30) is highly non-linear, however, and a general closed-form solution would not be expected anyway.

6. Solution for constant wall stress, $c = 0$

In the case $c = 0$, the first integration of (30) is trivial and the outer solution is the first of equations (32). The integrated wall-layer equation, the matching point condition, and the stress distribution in the outer layer are as follows:

$$(a) \quad s' + \lambda b r = K,$$

$$(b) \quad \eta_e r_e = \lambda b_e,$$

$$(c) \quad s = A\, e^{-\frac{1}{2}\eta^2}.$$

The constant of integration K can be found immediately by matching slopes at η_e:

$$\begin{aligned} s'_e &= -\eta_e r_e^2 && \text{by } (c), \\ &= -\lambda b_e r_e && \text{by } (b), \\ &= K - \lambda b_e r_e && \text{by } (a). \end{aligned}$$

Thus $K = 0$. Equation (a) becomes

$$2r' + \lambda b = 0,$$

or, written out in full,
$$2r' + \lambda + 2\lambda B\,(\eta r'/r) = 0,$$

where the fact that s' is negative has been anticipated. The equation can be inverted so that η becomes the dependent variable

$$\frac{d\eta}{dr}+\frac{2B}{r}\eta+\frac{2}{\lambda}=0.$$

The new equation is linear in η, and the solution satisfying the boundary condition $\eta(1)=0$ is

$$\eta=\frac{2}{\lambda(2B+1)}(r^{-2B}-r). \qquad (34)$$

The stress slope a appearing in (33) has the value $-\lambda$ for all B. If $B=0$, (34) can be written

$$r=1-\tfrac{1}{2}\lambda\eta.$$

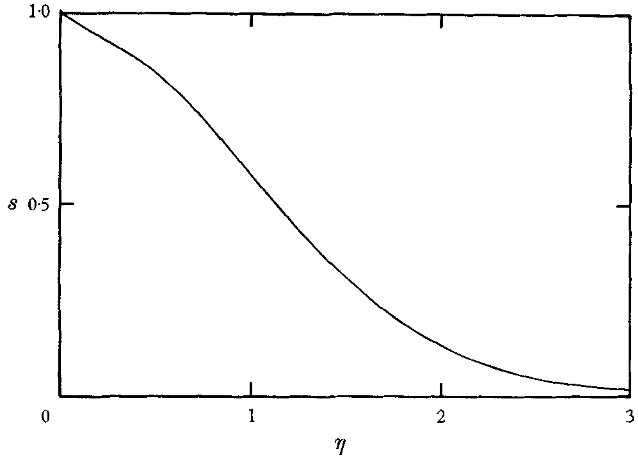

FIGURE 2. Stress profile in the case $c=0$.

B	a	η_e	A
0	−0·30	0·315	0·954
0·2	−0·30	0·309	0·954
0·5	−0·30	0·300	0·956
1·0	−0·30	0·287	0·958

TABLE 1

A, η_e and s_e can be computed by solving (b), (c) and equation (34) simultaneously at the matching point. Table 1 contains values of a, η_e and A obtained for $\lambda=0{\cdot}30$ and $B=0$, 0·2, 0·5 and 1·0. The stress profile for any of those values of B is plotted in figure 2. The profile is practically insensitive to Townsend's lateral transport term, because the turbulent energy density varies only slightly throughout the wall layer of a Rayleigh flow generated by a constant wall stress.

7. Solution for continuously separating flow, $c=-1$

The outer solution for the case $c=-1$ is the second of equations (32). Equation (30) for the wall layer is

$$\eta s''+\tfrac{1}{2}\lambda[(br)+\eta(br)']=0,$$

or

$$(\eta s'-s)'+\tfrac{1}{2}\lambda(\eta br)'=0.$$

The first integral of (30) satisfying $s(0) = 0$, the matching condition and the outer solution are as follows:

$$(a)\quad \eta s' - s + \tfrac{1}{2}\lambda\eta b r = 0,$$
$$(b)\quad \eta_e r_e = \lambda b_e,$$
$$(c)\quad s = A\eta\, e^{-\frac{1}{2}\eta^2}.$$

In this case, however, the condition that s' is continuous across the junction η_e is identically satisfied:

$$s'_e = \frac{s_e}{\eta_e} - \frac{\eta_e s_e}{2} \quad \text{by} \quad (c),$$
$$= \frac{s_e}{\eta_e} - \frac{\lambda}{2} b_e r_e \quad \text{by} \quad (b),$$

and the last equality holds identically for any solution of (a). The amplitude of the profile $s(\eta)$ appears to be non-unique.

The non-uniqueness is due to a physical circumstance peculiar to continuously separating flow. In every other case the momentum in the field is determined by the history of the stress at the wall, but the wall stress is zero for continuously separating flow, and the momentum is injected into the field by unspecified means. According to (22),

$$U = -\left(\frac{Q}{kt}\right)^{\frac{1}{2}} \frac{2s}{\eta}. \tag{35}$$

Thus

$$M = 2Q\int_0^\infty \frac{s}{\eta}\, d\eta$$

specifies the amount of negative momentum in the mean flow. Equation (27) implies that $M \to Q$ as $c \to -1$, but the equation itself breaks down when c is exactly -1, since u_τ is then zero. It is reasonable, nevertheless, to define $Q \equiv M$ when $c = -1$ in order to make the relation between M and Q continuous over the whole family of self-similar flows. That definition requires the stress profile to satisfy the side condition

$$\int_0^\infty \frac{s}{\eta}\, d\eta = \tfrac{1}{2}, \tag{36}$$

which removes the non-uniqueness.

Equation (a) is

$$2\eta r' - r + \tfrac{1}{2}\lambda\eta + \lambda B\eta^2 (r'/r) = 0,$$

written out in full under the anticipation that s' is positive. The boundary condition $s(0) = 0$ has already been used to perform the first integration. A second integration can be carried out in terms of the condition $s'(0) = a$. The substitutions

$$R = r/(a\eta)^{\frac{1}{2}}, \quad Y = (\eta/a)^{\frac{1}{2}}$$

lead to a linear equation for $Y(R)$:

$$\frac{dY}{dR} - \frac{B}{(1-B)R}\, Y + \frac{2}{\lambda(1-B)} = 0.$$

The solution satisfying $Y(1) = 0$ is

$$Y = \frac{2}{\lambda(1-2B)}\left(R^{B/(1-B)} - R\right). \tag{37}$$

The relation takes on simpler forms for some special values of B:

$$B = 0, \quad r = (a\eta)^{\frac{1}{2}} - \tfrac{1}{2}\lambda\eta;$$

$$B = \tfrac{1}{2}, \quad \eta = \frac{2}{\lambda} r \ln\left(\frac{a\eta}{s}\right);$$

$$B = 1, \quad r = (a\eta)^{\frac{1}{2}}.$$

The polynomial approximation

$$r \approx (a\eta)^{\frac{1}{2}} - \frac{\lambda(1-B)}{2}\eta$$

satisfies (37) very accurately everywhere in the wall layer for all reasonable values of B.

B	a	η_e	A
0	0·382	0·721	0·273
0·2	0·358	0·633	0·277
0·5	0·324	0·472	0·280
1·0	0·282	0	0·282

TABLE 2

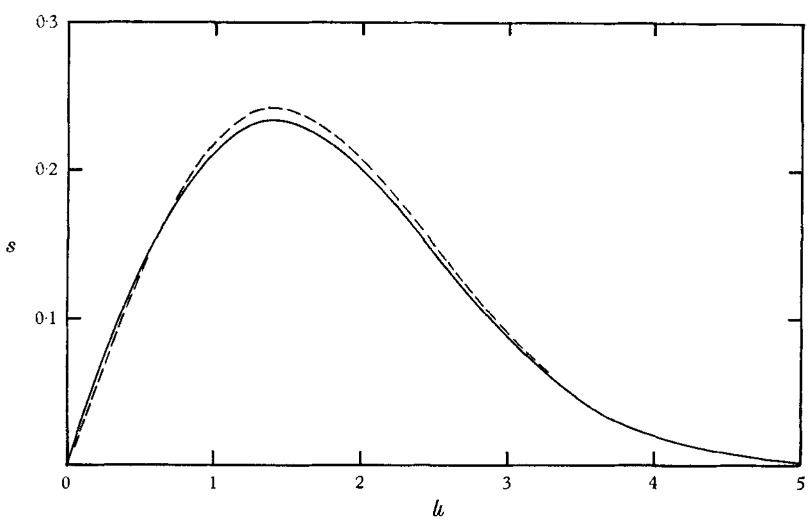

FIGURE 3. Stress profile in the case $c = -1$: ——, $B = 0$; ----, $B = 1$.

The quantities a, A, η_e and s_e can be found by solving (b), (c) and equation (37) simultaneously at the matching point and imposing the side condition (36). Results for $\lambda = 0{\cdot}30$ and the same values of B as before are presented in table 2. The theory is consistent only for values of B in the range $0 \leqslant B \leqslant 1$. If $B < 1$, $s(\eta)$ bends over fast enough in the wall layer to match the outer solution (c) in amplitude and slope at a point $\eta_e > 0$. If $B = 1$, s is linear in the wall layer and matches (c) only if $\eta_e = 0$. The wall layer disappears for $B \geqslant 1$. The stress profile $s(\eta)$ is plotted in figure 3 for the extreme values $B = 0$ and $B = 1$. The corresponding velocity profiles s/η (cf. equation (35)) are plotted in figure 4.

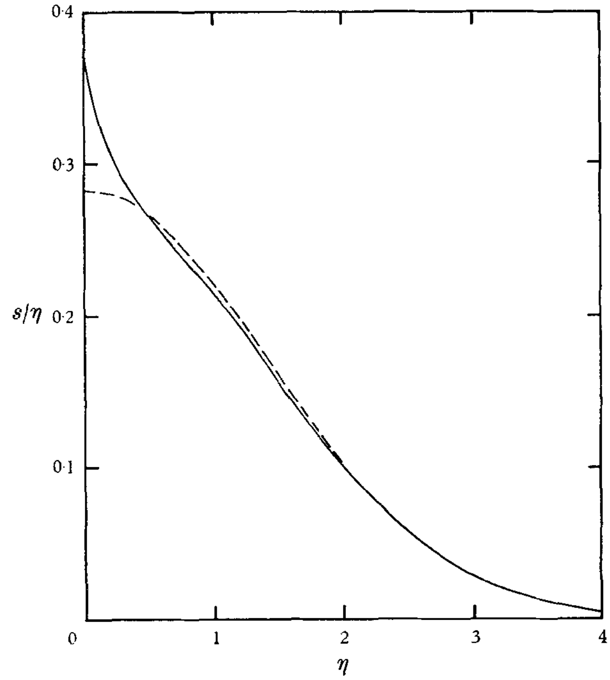

FIGURE 4. Velocity profile in the case $c = -1$: ——, $B = 0$; ----, $B = 1$.

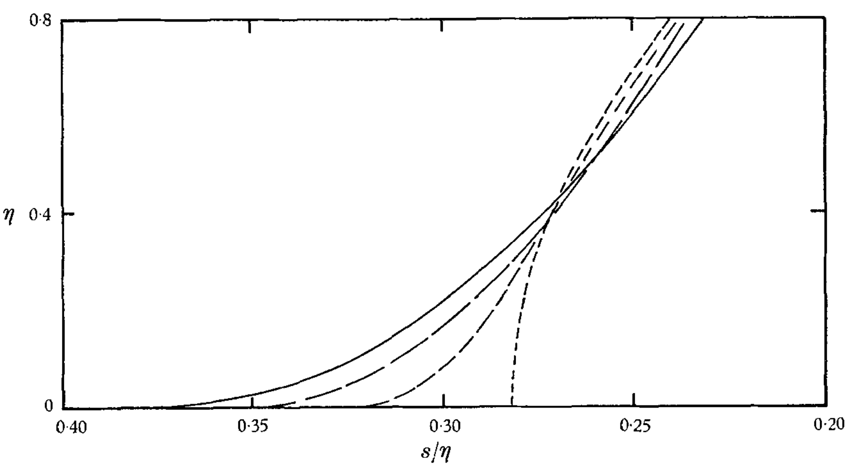

FIGURE 5. Velocity near the wall in the case $c = -1$: ——, $B = 0$; ——, $B = 0{\cdot}2$; ---, $B = 0{\cdot}5$; ····, $B = 1{\cdot}0$.

Figure 5 is a close-up plot of the velocity profile near the wall for various values of B. The figure is oriented so that the abscissa represents the wall. The shape of the profile near the wall depends sensitively on the value assigned to the coefficient B of Townsend's lateral transport term. Lateral transport of turbulent energy can have an important effect on the mean flow in a separating layer, Mellor & Gibson adopted the Prandtl relation (1) for their boundary-layer calculations, and thus, in effect, set $B = 0$. If Townsend's relation is an improvement, then Stratford's experimental data for the continuously separating boundary layer should deviate from Mellor & Gibson's calculated profile in the same way that profiles for successively higher values of B deviate from the $B = 0$ profile in figure 5. A comparison between figure 5 and Mellor & Gibson's figure 9 shows that Townsend's transport term modifies the predicted velocity profile in just the right way and that $B \approx 0{\cdot}2$ is a reasonable value. The comparison is only qualitative, of course, since Mellor & Gibson take into account advection terms that are absent in the Rayleigh problem.

According to (23) or (35), the wall-speed programme

$$U_w = -2a\left(\frac{M}{k(t-t_0)}\right)^{\frac{1}{2}}$$

maintains a continuously separating Rayleigh layer containing momentum $-M$. A velocity slip across the laminar sublayer, negligible for large values of the Reynolds number M/ν, has been disregarded (Stratford 1959).

8. Concluding remarks

It would be very difficult to establish a turbulent Rayleigh layer experimentally. Any practical set-up would involve approximating an infinite plane wall with a curved or finite surface. Departures from the ideal geometry would exert an unpredictable influence on the flow. Conceptually, however, a Rayleigh experiment is simple: to generate a self-similar Rayleigh layer, for example, move a plate according to the programme (23). The connexion between the similarity solutions and the physical situation is very clear. Two-dimensional boundary layers have their own difficulties: practical experiments can be carried out, but the theory is cumbersome. Boundary and Rayleigh layers share a common eddy structure; the differences arise in the kinematics of the mean flow. The two kinds of flow are so alike that self-similar Rayleigh profiles are related to equilibrium boundary-layer profiles by a formal analogy that becomes quantitatively exact in the limit of infinite Reynolds number. The Rayleigh problem should therefore be a useful touchstone for theories of Reynolds stress. It proved to be a natural setting for Townsend's energy-equilibrium theory.

I enjoyed much stimulating discussion about Rayleigh flow with Peter Bradshaw at the National Physical Laboratory. The major part of this research was carried out there during the summer of 1964. The work was completed under the auspices of the United States Atomic Energy Commission at the Lawrence Radiation Laboratory, Livermore, California.

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